copo

1 a) ① $f(x) = e^{x-2} - x^2$ $f'(x) = e^{x-2} - 2x$

Newton $x_{n+1} = x_n - \dfrac{e^{x_n-2} - x_n^2}{e^{x_n-2} - 2x_n}$

② pro: fast, 2nd order convergence, no α tuning, easy program
con: $f'(x)$ needed, divide by zero risk,
2 function evals per iteration

b) $g(x) = x + \alpha(e^{x-2} - x^2)$ $x_{n+1} = g(x_n)$
$g'(x) = 1 + \alpha(e^{x-2} - 2x)$
$g'(\frac{1}{2}) = 1 + \alpha(e^{-3/2} - 1) = 0$ $\alpha = \dfrac{-1}{e^{-3/2} - 1} = 1.287\ldots$

c) ① $g(x) = \sqrt{e^{x-2}}$ $g'(x) = \frac{1}{2}(e^{x-2})^{-\frac{1}{2}} \cdot e^{x-2} = \frac{1}{2}\sqrt{e^{x-2}}$

$g'(\frac{1}{2}) = \frac{1}{2}e^{-3/4} = 0.23628 < 1$
$\Rightarrow$ conv

g' near sol. < 1 $\Rightarrow$ conv

② $\varepsilon_4 \leq \dfrac{K}{1-K}|x_3 - x_4|$ $K \simeq \dfrac{x_4 - x_3}{x_3 - x_2}$

$K = 0.23209855$ $\varepsilon_4 \leq 1.05748 \; 10^{-4}$

③ $\text{Steff} = x_4 - \dfrac{(x_4 - x_3)^2}{(x_4 - x_3) - (x_3 - x_2)} = 0.46392184$

d) initialisation
loop construction
iteration formula
error estimate in stop crit

2a) ① [0 2] 5 segments 3^{rd} segment [0.8 1.2]

$$\text{trap} = \frac{0.4}{2}\left(f(0.8)+f(1.2)\right) = 0.2872133$$

② not optimal, because $f''(x) \to \infty$ when $x \to 0$

b) ① $$q = \left|\frac{I_{64} - I_{32}}{I_{128} - I_{64}}\right| = 4.00010923$$

$q \approx 4 \to 2^{nd}$ order o.k. $\to$ error est allowed

$$\mathcal{E}_{128} \leq \frac{|I_{128} - I_{64}|}{3} = 1.220666\ 10^{-4}$$

② $$\mathcal{E}_{glob} \leq (1-0)\, M \left(\frac{1}{128}\right)^2 \frac{1}{12}$$

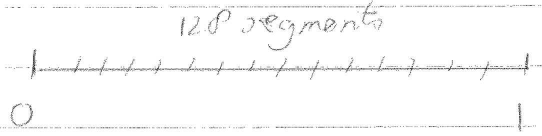
$$f(x) = 8x^3 \quad f''(x) = 48x \Rightarrow M = 48$$

$$\mathcal{E}_{glob} \leq 1 \times 48 \left(\frac{1}{128}\right)^2 \frac{1}{12} = 2.44140625\ 10^{-4}$$

③ $\mathcal{E}_{128}$ better, since $\mathcal{E}_{glob}$ based on max on [0,1]
which is only reached at $x=1$
f'' "much" smaller at other $x \neq 1$

c) ① $$T_2 = \frac{I_{128} - I_{64}}{3} + I_{128} = 2.0000000133$$

② all function evals for I_{128} also needed for I_{64}



128 grid 129 gridpoints
129 evaluations

d) grid
loop construction
trapezium formula

refinement procedure
error estimate in stop crit

cupo

3 a) $y' = -y + x \qquad y(0)=1$

(1) $y_1 = y_0 + h(-y_0 + x_0)$
$= 1 + 0.25(-1+0) = 0.75$

$y_2 = y_1 + h(-y_1 + x_1)$
$= 0.75 + 0.25(-0.75 + 0.25)$
$= 0.625$

(2) $y_1 = y_0 + h(-y_1 + x_1)$

$(1+h)y_1 = y_0 + hx_1 \qquad (1+0.5)y_1 = 1 + 0.5 \times 0.5$

$y_1 = 0.833333$

b) $y_{n+1} = y(x_{n+1}) + \alpha_3 h^3 + \alpha_4 h^4$
$\hat{y}_{n+1} = y(x_{n+1}) + \alpha_3 8h^3 + \alpha_4 16h^4$

$\Rightarrow \frac{\hat{y}_{n+1} - y_{n+1}}{7} = \alpha_3 h^3 + O(h^4)$

$\to$ Error estimate $\left|\frac{0.243890 - 0.248521}{7}\right|$
$= 6.61571 \cdot 10^{-4}$

$y(x) = y_h(x) + \alpha_3 h^3 + \alpha_4 h^4$
$y(x) = y_{h/2}(x) + \alpha_3 \frac{h^3}{8} + \alpha_4 \frac{h^4}{16}$

$\Rightarrow 7y(x) = 8y_{h/2}(x) - y_h(x) + O(h^4)$
$\Rightarrow y(x) = \frac{8}{7} y_{h/2}(x) - \frac{1}{7} y_h(x) + O(h^4) \to$ extrapolation

$\frac{8}{7}\, 0.248521 - \frac{1}{7}\, 0.243890 = 0.249183$

c) Expl. $y_{n+1} = y_n + h f(x_n, y_n)$
Impl $y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$
take $\frac{1}{2}$ Expl + $\frac{1}{2}$ Impl
combination $\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$

d) grid
loop construction
Euler formula
refinement procedure full grid
error estimate (full grid) in step crit

copo

4 a) $y''(x) + e^{x} y(x) = x^2$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + e^{x_i} y_i = x_i^2$$

N-1 segments $\Rightarrow$ interior points 2 -- N-1

bound cond $y(0) = 1 \quad y_1 = 1$

$y(4) = 2 \quad y_N = 2$

$$\left[\begin{array}{ccccccc|c}
1 & & & & & & & 1 \\
\frac{1}{\Delta x^2} & \frac{-2}{\Delta x^2} + e^{x_1} & \frac{1}{\Delta x^2} & & & & & x_1^2 \\
 & \ddots & \ddots & \ddots & & & & \\
 & & L_i = \frac{1}{\Delta x^2} & D_i = \frac{-2}{\Delta x^2} + e^{x_i} & R_i = \frac{1}{\Delta x^2} & & & x_i^2 = b_i \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & \frac{1}{\Delta x^2} & \frac{-2}{\Delta x^2} + e^{x_N} & \frac{1}{\Delta x^2} & x_N^2 \\
 & & & & & & 1 & 2
\end{array}\right]$$

b) $y'(4) = 1 \quad \frac{y_N - y_{N-1}}{\Delta x} = 1 \quad \Rightarrow$ change in final row N:

$$\left[\begin{array}{ccc|c} \cdots\cdots & -\frac{1}{\Delta x} & \frac{1}{\Delta x} & 1 \end{array}\right]$$